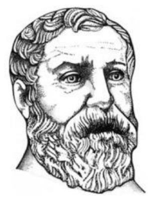
3. PAPPUS’ THEOREM

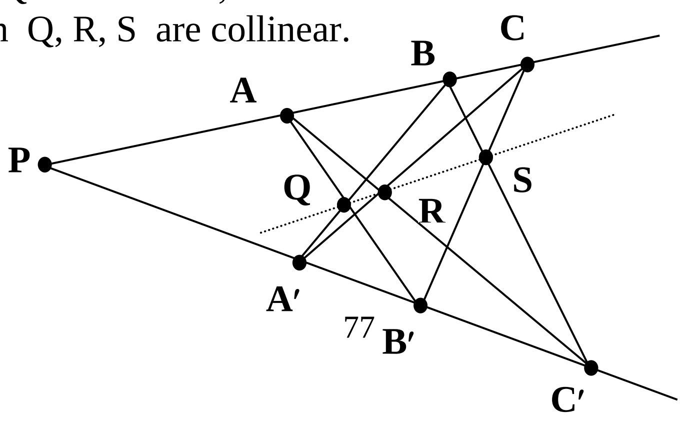
§3.1. Pappus’ Theorem

Pappus of Alexandria (c. 300 A.D.) was a Greek mathematician. We don’t seem to have an authentic picture of him. I found this on the web, claiming it to be of Pappus, but I found other places where the same face claimed to be other mathematicians from Alexandria. Still, it looks like an ancient geometer!



Pappus provided a particularly simple proof of the equality of the base angles of an isosceles triangle. His great work *A Mathematical Collection* is an important source of information about ancient Greek mathematics. Here’s his famous theorem, which he proved for the Affine Plane. Actually it’s a theorem of Projective Geometry.

Theorem 2 (PAPPUS): Suppose {A, B, C} and {A′, B′, C′} are two collinear sets where the 6 points are distinct and the two lines are distinct.
Let Q = AB′∩A′B, R = AC′∩A′C and S = BC′∩B′C.
Then Q, R, S are collinear.



Proof: Let $P = AB \cap A'B'$. The theorem is trivial if P coincides with any of the six points so we may assume that it is distinct from each of them.

By the Collinearity Lemma we may choose vectors **a**, **a**′ and scalars λ, λ' such that:

$$P = \langle \mathbf{p} \rangle;$$
$$A = \langle \mathbf{a} \rangle,\ A' = \langle \mathbf{a}' \rangle;$$
$$B = \langle \mathbf{p} + \mathbf{a} \rangle,\ B' = \langle \mathbf{p} + \mathbf{a}' \rangle;$$
$$C = \langle \lambda\mathbf{p} + \mathbf{a} \rangle,\ C' = \langle \lambda'\mathbf{p} + \mathbf{a}' \rangle.$$

Let $\mathbf{q} = (\mathbf{p} + \mathbf{a}) + \mathbf{a}' = (\mathbf{p} + \mathbf{a}') + \mathbf{a}$
$$\in (A' + B) \cap (A + B').$$
Hence $Q = \langle \mathbf{q} \rangle$.

Let
$$\begin{aligned}\mathbf{r} &= \lambda\lambda'\mathbf{p} + \lambda'\mathbf{a} + \lambda\mathbf{a}' \\ &= \lambda(\lambda'\mathbf{p} + \mathbf{a}) + \lambda'\mathbf{a} \\ &= \lambda'(\lambda\mathbf{p} + \mathbf{a}) + \lambda\mathbf{a}' \\ &\in (A + C') \cap (A' + C).\end{aligned}$$
Hence $R = \langle \mathbf{r} \rangle$.

Finally let:
$$\begin{aligned}\mathbf{s} &= \mathbf{q} - \mathbf{r} \\ &= \mathbf{p} + \mathbf{a} + \mathbf{a}' - \lambda\lambda'\mathbf{p} - \lambda'\mathbf{a} - \lambda\mathbf{a}' \\ &= (1 - \lambda')(\mathbf{p} + \mathbf{a}) + (1 - \lambda)(\lambda'\mathbf{p} + \mathbf{a}') \in B + C' \\ &= (1 - \lambda)(\mathbf{p} + \mathbf{a}') + (1 - \lambda')(\lambda\mathbf{p} + \mathbf{a}) \in B' + C.\end{aligned}$$
Hence $S = \langle \mathbf{s} \rangle$.

Since $\mathbf{q} = \mathbf{r} + \mathbf{s}$ it follows that **q**, **r**, **s** are linearly dependent and so Q, R, S are collinear.

§3.2. Finite Projective Planes

We denote the projective plane that's formed from the field F by the symbol $\wp$**(F)**. So the real projective plane is $\wp(\mathbb{R})$. Here we'll consider projective planes where the field is finite.

The simplest finite fields are the integers modulo a prime. The field of integers modulo p is denoted by $\mathbb{Z}_p$. The smallest field is $\mathbb{Z}_2$, the field with 2 elements. These elements are written 0, 1 and addition and multiplication can be described by the tables:

+	**0**	**1**
0	0	1
1	1	0

×	**0**	**1**
0	0	0
1	0	1

Let's examine the smallest projective plane, $(\mathbb{Z}_2)$. The field $\mathbb{Z}_2$ has 2 elements. The vector space $\mathbb{Z}_2^3$ has $2^3 = 8$ elements. Of these one, the zero vector (0, 0, 0), doesn't span a projective point. The other 7 do. Moreover, because there's only one non-zero scalar, each of these 7 vectors spans a different projective point. So $\wp(\mathbb{Z}_2)$ has 7 points:

$$\langle(0, 0, 1)\rangle,\ \langle(0, 1, 0)\rangle,\ \langle(0, 1, 1)\rangle,\ \langle(1, 0, 0)\rangle,$$
$$\langle(1, 0, 1)\rangle,\ \langle(1, 1, 0)\rangle,\ \langle(1, 1, 1)\rangle.$$

By duality there must be the same number of projective lines:

$$\langle(0, 0, 1)\rangle^{\perp},\ \langle(0, 1, 0)\rangle^{\perp},\ \langle(0, 1, 1)\rangle^{\perp},\ \langle(1, 0, 0)\rangle^{\perp},$$
$$\langle(1, 0, 1)\rangle^{\perp},\ \langle(1, 1, 0)\rangle^{\perp},\ \langle(1, 1, 1)\rangle^{\perp}.$$

As an example, the projective point $\langle(0, 1, 1)\rangle$ lies on the projective line $\langle(1, 1, 1)\rangle^{\perp}$ since

$$(0, 1, 1).(1, 1, 1) = 0 + 1 + 1 = 0.$$

How many points are there on each of these 7 lines? A projective line is a 2-dimensional subspace and so has $2^2 = 4$ vectors. Of these, the zero vector doesn't span a projective point but the other three do. They span three distinct projective points. So on each line in $\wp(\mathbb{Z}_2)$ we have three points. And how many lines through each point? That's easy – by duality that's the same number, three.

Now it's very messy working with these vectors and having to do a modulo 2 dot product every time we want to see if a point lies on a line. It's much easier to code them and to construct a Collinearity Table.

We code the points as A, B, C, D, E, F, G and the lines as *a*, *b*, *c*, *d*, *e*, *f*, *g*.

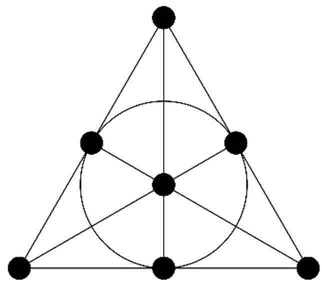
$$
\begin{aligned}
A &= \langle(0, 0, 1)\rangle, & a &= \langle(0, 0, 1)\rangle^{\perp};\\
B &= \langle(0, 1, 0)\rangle, & b &= \langle(0, 1, 0)\rangle^{\perp};\\
C &= \langle(0, 1, 1)\rangle, & c &= \langle(0, 1, 1)\rangle^{\perp};\\
D &= \langle(1, 0, 0)\rangle, & d &= \langle(1, 0, 0)\rangle^{\perp};\\
E &= \langle(1, 0, 1)\rangle, & e &= \langle(1, 0, 1)\rangle^{\perp};\\
F &= \langle(1, 1, 0)\rangle, & f &= \langle(1, 1, 0)\rangle^{\perp};\\
G &= \langle(1, 1, 1)\rangle. & g &= \langle(1, 1, 1)\rangle^{\perp}.
\end{aligned}
$$

Now, with a certain amount of calculation we can work out which points lie on which line and so produce the following Collinearity Table.

a	*b*	*c*	*d*	*e*	*f*	*g*
B	A	C	A	B	A	C
D	D	D	B	E	F	E
F	E	G	C	G	G	F

The columns of this table list the seven lines, and for each one, the three points on the line. Look closely at the table. Any two capital letters occur together in exactly one column (any two points lie on exactly one line) and any two columns have exactly one capital letter in common (any two lines intersect in exactly one point).

Is it possible to draw a picture of the 7 point projective plane? The following diagram is the best we can do. But one of the 'lines' is actually a circle. This is unavoidable because it's not possible to embed $\wp(\mathbb{Z}_2)$ in $\mathbb{R}^3$. Can you label the vertices in this diagram so that the 7 lines correspond to the 7 lines in the above table?



There's no longer any need to go back to the original vectors. Every geometric question for $\wp(\mathbb{Z}_2)$ can be answered by just examining this table (or your labeled diagram).

Question 1: Does B lie on the line g?
Answer: No, since B isn't in column g.

Question 2: What is $c \cap f$?
Answer: G, since only G is in both columns.

Question 3: What is the line AD?
Answer: ***b*** since that's the column where both A and D occur.

Question 4: Are the points A, F, G collinear?
Answer: Yes, since they all lie on the line f.

Question 5: Are the lines b, c, g concurrent?
Answer: No, since there's no letter common to all three columns.

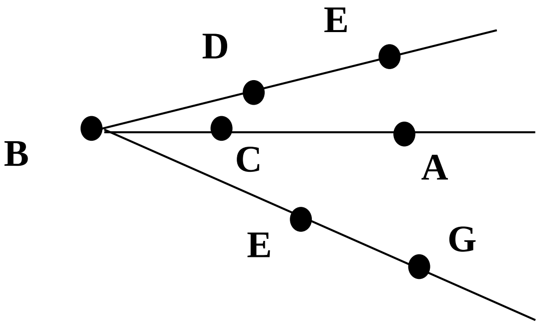
Question 6: Are the triangles ΔCDE and ΔAFG in perspective from a point?
Answer: Yes, they're in perspective from B since:

{B, D, F}, {B, C, A} and {B, E, G}

are three sets of collinear points.

Question 7: Are the triangles ΔCDE and ΔAFG in perspective from a line?
Answer: Yes, by question 6 and Desargues' Theorem.

Question 8: Which line?
Answer: CD ∩ AF = $c \cap f$ = G
CE ∩ AG = $g \cap f$ = F
DE ∩ FG = $b \cap f$ = A.
Note that A, F, G are collinear, as predicted by Desargues' Theorem. From the table we see that this line is *f*. So the triangles are in perspective from the line ***f***.



In the above triangle no attempt has been made to put the points 'in the right places' – 'right places' doesn't make sense for the 7-point projective plane. In fact we've had to place E in two places.

Note that there's a certain amount of degeneracy in this example in that the triangle A, F, G consists of 3 collinear points. It's impossible in this tiny example, with only seven points, to avoid getting a degenerate triangle. Nevertheless Desargues' Theorem still works even if one of the triangles is degenerate.

Question 9: Which of the seven lines is the ideal line?
Answer: This is a non-question. Remember that any line can be considered to be the ideal line by placing the

relevant affine plane appropriately. For example if we want *d* to be the ideal line then its three points A, B, C will become the ideal points. Removing the line *d* and the three points A, B, C we are left with the four 'ordinary' points D, E, F, G and the six 'ordinary' lines *a*, *b*, *c*, *e*, *f*, *g*. There are two ordinary points on each ordinary line, as follows:

a	***b***	***c***	***e***	***f***	***g***
D	D	D	E	F	E
F	E	G	G	G	F

This is the Affine Plane over $\mathbb{Z}_2$. Now we have lines which don't intersect, and which are therefore considered to be parallel. For example $a \parallel e$. These lines don't intersect in an ordinary point, but, going back to the original table, they intersect in the ideal point B.

It must be emphasized that this distinction between ideal and ordinary points depended on our arbitrary choice of *d* as ideal line. Any line could have been chosen and whether a point is ordinary or ideal would change. In the projective plane itself there's no such distinction.

We now consider the projective plane $\wp(\mathbb{Z}_3)$ over the field of integers modulo 3. This field has the following addition and multiplication tables:

+	**0**	**1**	**2**
0	0	1	2
1	1	2	0
2	2	0	1

×	**0**	**1**	**2**
0	0	0	0
1	0	1	2
2	0	2	1

There will now be $3^3 = 27$ vectors. Removing the zero vector we are left with 26 non-zero vectors. But because there are now 2 non-zero scalars each projective point will be spanned by 2 different vectors. So the 26 non-zero vectors in fact only give us 26/2 = 13 distinct points. And, by duality, there must be 13 lines. Each line is a 2-dimensional subspace containing $3^2 = 9$ vectors, of which 8 are non-zero. But these will only give us 8/2 = 4 points on the line because there are 2 non-zero scalars and $\langle 2\mathbf{v} \rangle = \langle \mathbf{v} \rangle$.

So the 13-point projective plane has 13 points and 13 lines with 4 points on each line and 4 lines through each point.

By working through the process of constructing finite projective planes one can obtain, by suitably coding the 1-dimensional subspaces, the following collinearity table. Just for fun we'll use the names of the 13 cards in a suit in a pack of cards to denote the 13 points and the same labels for the 13 lines. The entries at the top of the columns denote the lines and the entries in the body of the table give the 4 points on that line.

A	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**J**	**Q**	**K**
9	7	Q	Q	J	10	K	Q	J	A	K	5	4
J	6	8	10	8	A	8	9	10	Q	10	K	A
7	5	5	7	6	8	7	6	5	J	6	9	2
4	A	4	3	3	9	2	2	2	K	4	3	3

Note that every pair of ‘points’ occurs in exactly one column and that every pair of ‘lines’ has exactly one common point.

The next smallest projective plane is the 21-point plane that arises from the field with 4 elements. But this field is not $\mathbb{Z}_4$ because 4 isn’t prime. It’s denoted by the symbol GF(4) (and is also called a ‘Galois field’ after the famous mathematician Galois).

It arises by starting with the field $\mathbb{Z}_2$ and adjoining to it a symbol x that satisfies the rule that $x^2 = x + 1$. There are four elements 0, 1, x and $x + 1$. (Any higher powers are unnecessary because $x^2 = x + 1$.)

Let $y = x + 1$. The addition and multiplication tables for GF(4) are as follows. Note that $y^2 = (x + 1)^2$

$$= x^2 + 2x + 1 = x^2 + 1 = (x + 1) + 1 = x.$$

THE FIELD GF(4)

+	**0**	**1**	***x***	***y***
0	0	1	x	y
1	1	0	y	x
x	x	y	0	1
y	y	x	1	0

×	**0**	**1**	***x***	***y***
0	0	0	0	0
1	0	1	x	y
x	0	x	y	1
y	0	y	1	x

We can code *x* and *y* as 2, 3 respectively to give:

THE FIELD GF(4)

+	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

×	**0**	**1**	**2**	**3**
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

§3.3. Combinatorial Applications

Finite projective planes are useful in certain combinatorial situations. The following is a very simple example where such a combinatorial problem can be solved.

Example 3: A chemist shop operates 12 hours a day 7 days a week. It needs to have 3 people on duty on any given day and, because of the long days, they each work 3 days a week. Now it's desirable, for the sake of continuity, that at least one of the three people on duty on any given day was one of the three working on any other day (so that a customer could come in on Friday, for example, and say "I had this prescription dispensed last Tuesday" and one of the Friday staff would know what she was talking about, having worked on the Tuesday). Draw up such a roster.

Solution: With 7 days, and 3 people required each day, there are 21 slots to be filled on a weekly roster. With a 3 day working week it's clear that exactly 7 staff are needed

– provided we could meet the requirement that the staff for any 2 days have at least one person in common.

Well, of course, we simply assign each member of staff to a point in the 7-point projective plane. The staff roster for the 7 days can be obtained by simply taking the 7 projective lines. So the table above for the 7-point projective plane will give a suitable roster. (Simply allocate the 7 labels A to G to the staff and the 7 labels a to g to the days of the week.)

EXERCISES FOR CHAPTER 3

Exercise 1: Write out, with diagrams, at least six Euclidean interpretations of Pappus' Theorem.

Exercise 2: State the dual of Pappus' Theorem and illustrate it with a diagram.

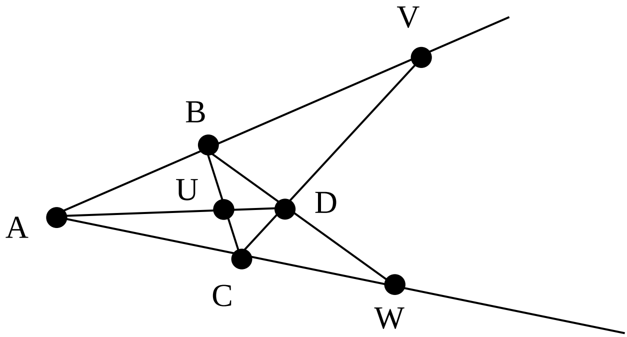
Exercise 3: In $\wp(\mathbb{Z}_7)$, let A = $\langle(1, 2, 3)\rangle$, B = $\langle(1, 1, 4)\rangle$, C = $\langle(5, 0, 1)\rangle$, D = $\langle(1, 1, 1)\rangle$. Find AB $\cap$ CD.

Exercise 4: If F is the field $\mathbb{Z}_p$ of integers modulo p how many points and how many lines, are there in $\wp$(F)? How many points lie on each line? How many lines pass through each point?

Exercise 5:

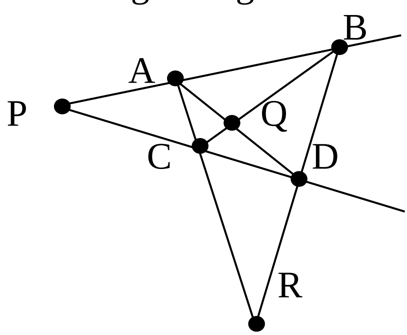
Let A, B, C and D be four points in the real projective plane, no three of which are collinear. Let U, V and W

denote the points of intersection of various lines as indicated in the diagram.
Suppose that A, B, C, V and W have vector representations A = $\langle \mathbf{a} \rangle$, B = $\langle \mathbf{b} \rangle$, C = $\langle \mathbf{c} \rangle$,
V = $\langle \mathbf{a} + \mathbf{b} \rangle$ and W = $\langle \mathbf{a} + \mathbf{c} \rangle$.



(i) Show that D = $\langle \mathbf{a} + \mathbf{b} + \mathbf{c} \rangle$ and find a vector for U.
(ii) Show that U, V and W are never collinear in $\wp(\mathbb{R})$, whereas they are always collinear in $\wp(\mathbb{Z}_2)$. What happens in $\wp(\mathbb{Z}_3)$?

Exercise 6: Suppose that P, A, C are three non-collinear points in the following configuration.



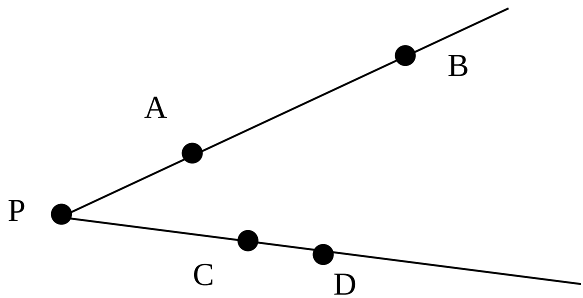
Let P = $\langle \mathbf{p} \rangle$, A = $\langle \mathbf{a} \rangle$, B = $\langle \mathbf{p} + \mathbf{a} \rangle$, C = $\langle \mathbf{c} \rangle$, D = $\langle \mathbf{p} + \mathbf{c} \rangle$ and let Q = AD $\cap$ BC, R = AC $\cap$ BD.
(i) Explain why R = $\langle \mathbf{a} - \mathbf{c} \rangle$ and Q = $\langle \mathbf{p} + \mathbf{a} + \mathbf{c} \rangle$.

(ii) Prove that if F is the field of real numbers then P, Q, R cannot be collinear.

(iii) Find a field F in which P, Q, R are collinear.

(iv) Let S = PR $\cap$ BC, T = AS $\cap$ BR, U = AR $\cap$ PQ.
Write down S, T, U in terms of **p**, **a**, **c**.
(v) Prove that if F is the field $\mathbb{Z}_3$ then P, U, T are collinear.

Exercise 8:
In the following diagram in $\wp(\mathbb{R})$, let P = $\langle \mathbf{p} \rangle$, A = $\langle \mathbf{a} \rangle$, B = $\langle \mathbf{p} + \mathbf{a} \rangle$, C = $\langle \mathbf{c} \rangle$, D = $\langle \mathbf{p} + \mathbf{c} \rangle$.
Construct the points $\langle \mathbf{p} + 2\mathbf{a} \rangle$ and $\langle \mathbf{p} + (3/2)\mathbf{a} \rangle$.

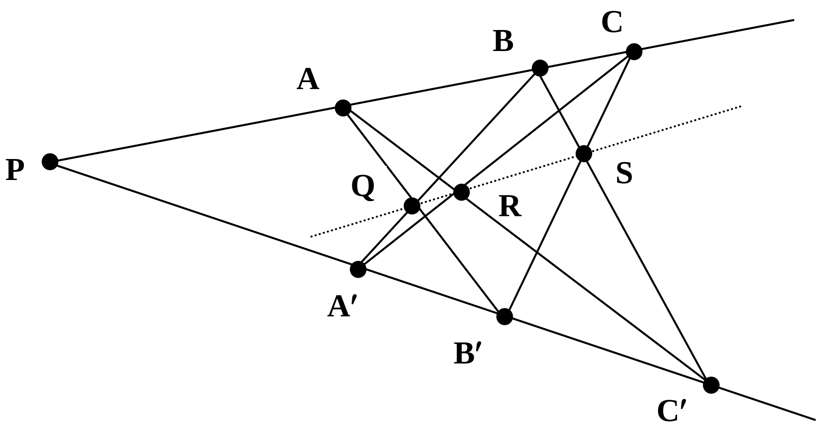


Exercise 9: In $\wp(\mathbb{Z}_5)$, let P = $\langle \mathbf{p} \rangle$, A = $\langle \mathbf{a} \rangle$, B = $\langle \mathbf{a} + \mathbf{c} \rangle$, C = $\langle \mathbf{c} \rangle$, D = $\langle \mathbf{p} + \mathbf{c} \rangle$.
Give a construction (that is a sequence of steps expressing new points as intersections of lines through previously constructed points) for $\langle \mathbf{p} + 3\mathbf{a} \rangle$, as a sequence of line intersections. (Don't attempt to draw the construction.)

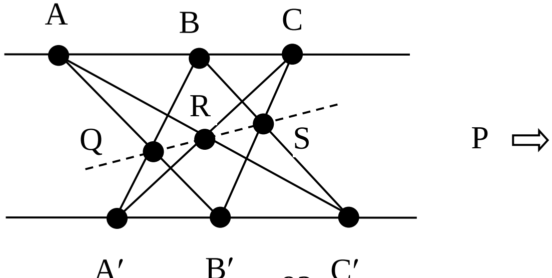
SOLUTIONS TO CHAPTER 3

Exercise 1:

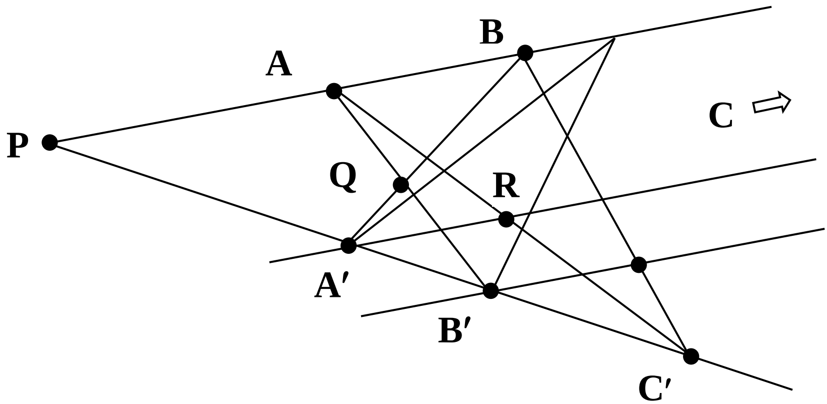
(1) Suppose that P, A, B, C, A′, B′, C′ are distinct points with P, A, B, C and P, A′, B′, C′ forming two distinct lines.
Let Q = A ′B ∩ AB ′, R = A ′C ∩ AC ′ and
S = B ′C ∩ BC ′. Then Q, R, S are collinear.



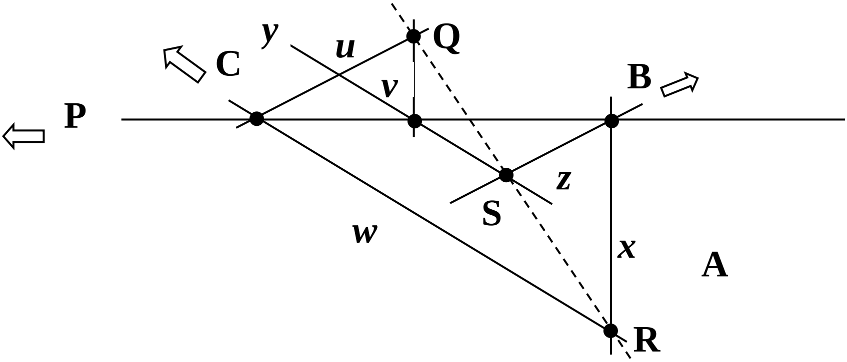
(2) Suppose that A, B, C, A′, B′, C′ are distinct points with A, B, C and A′, B′, C′ forming two distinct lines and with ABC parallel to A′B′C′.
Let Q = A′B∩AB′, R = A′C∩AC′ and S = B′C∩BC′.
Then Q, R, S are collinear.



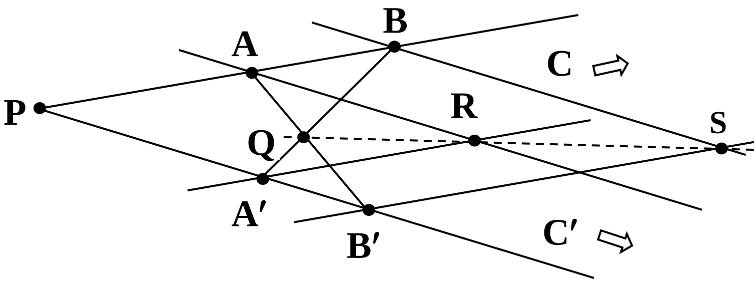
(3) Suppose that P, A, B, A′, B′, C′ are distinct points with P, A, B and P, A′, B′, C′ forming distinct lines. Let *m* be the line through A′ parallel to PAB and let *n* be the line through B′ parallel to PAB.
Let Q = A′B∩AB′, R = *m*∩AC′ and S = *n*∩BC′.
Then Q, R, S are collinear.



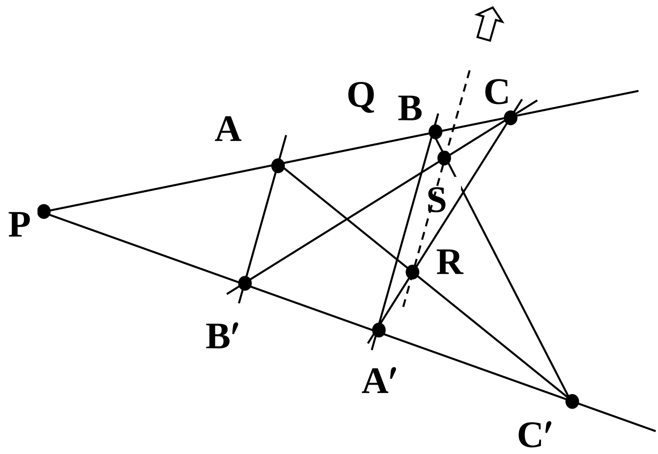
(4) Suppose that A′, B′, C′ are distinct points with A′, B′, C′ collinear. Let *m*, *n*, *r* be lines such that no two of them are parallel. Let *u* be the line through A′ parallel to *r*. Let *v* be the line through B′ parallel to *n*. Let *w* be the line through A′ parallel to *m*. Let *x* be the line through C′ parallel to *n*. Let *y* be the line through B′ parallel to *m*. Let *z* be the line through C′ parallel to *r*. Let Q = *u*∩*v*, R = *w*∩*x* and S = *y*∩*z*. Then Q, R, S are collinear.



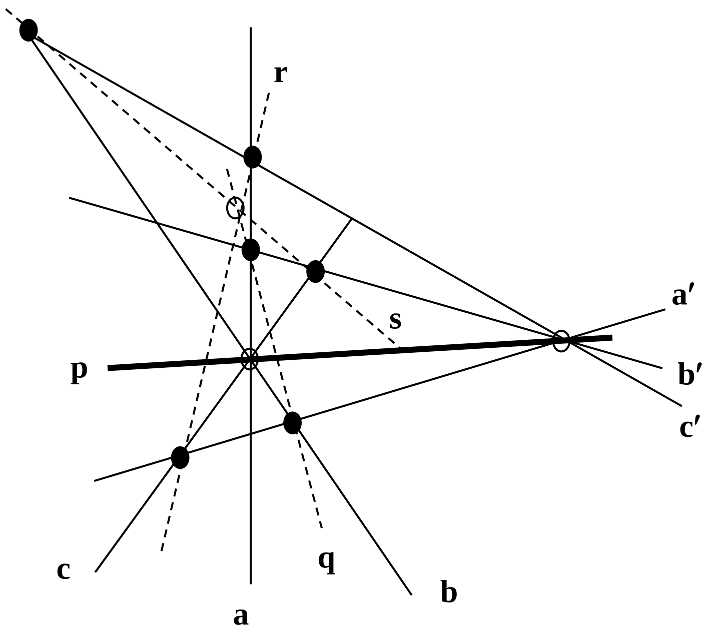
(5) Suppose P, A, B, A′, B′ are distinct points with P, A, B and P, A′, B′ forming distinct lines. Let *m* be the line through A parallel to PA′B′ and let *n* be the line through A′ parallel to PAB. Let *u* be the line through B parallel to PA′B′ and let *v* be the line through B′ parallel to PAB. Let Q = A′B∩AB′, R = $m \cap n$ and S = $u \cap v$. Then Q, R, S are collinear.



(6) Suppose P, A, B, C, A′, B′, C′ are distinct points with P, A, B, C and P, A′, B′, C′ forming distinct lines.
Suppose AB′ is parallel to A′B.
Let R = A′C ∩ AC′ and S = B′C ∩ BC′.
Then RS is parallel to AB′.



Exercise 2:



Suppose $\{a, b, c\}$ and $\{a', b', c'\}$ are two concurrent sets of lines where the 6 lines are distinct and the two points of concurrency are distinct.
Let $q = (a \cap b')(a' \cap b)$, $r = (a \cap c')(a' \cap c)$ and
$s = (b \cap c')(b' \cap c)$.Then q, r, s are concurrent.

Exercise 3: Although cross products only have a true geometric interpretation in $\mathbb{R}^3$, algebraically they give a vector orthogonal to two given vectors in F^3 for any field, provided we do the arithmetic in that field.
$AB = \langle(1, 2, 3) \times (1, 1, 4)\rangle^\perp = \langle(5, 6, 6)\rangle^\perp$ and $CD = \langle(5, 0, 1) \times (1, 1, 1)\rangle^\perp = \langle(6, 3, 5)\rangle^\perp$.
Hence $AB \cap CD = \langle(5, 6, 6) \times (6, 3, 5)\rangle = \langle(5, 4, 0)\rangle$.

Exercise 4: There are p^3 vectors in $\mathbb{Z}_p^3$. Of these $p^3 - 1$ are non-zero. Since there are
$p - 1$ non-zero scalars a given 1-dimensional subspace is spanned by any one of $p - 1$ non-zero vectors.
So the number of distinct 1-dimensional subspaces is

$$\frac{p^3 - 1}{p - 1} = p^2 + p + 1.$$

So $\wp(\mathbb{Z}_p)$ has $p^2 + p + 1$ points and, by duality, $p^2 + p + 1$ lines.
Each line is a 2-dimensional subspace containing $p^2 - 1$ non-zero vectors and hence, by the above argument,
$\frac{p^2 - 1}{p - 1} = p + 1$ one-dimensional subspaces. Thus each point has $p + 1$ lines passing through it.

Exercise 5:

(i) $D = BW \cap CV$. Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{b} + (\mathbf{a} + \mathbf{c})$, $\langle \mathbf{a} + \mathbf{b} + \mathbf{c} \rangle \in BW$.

Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{c} + (\mathbf{a} + \mathbf{b})$, $\langle \mathbf{a} + \mathbf{b} + \mathbf{c} \rangle \in CV$.

Hence it is $BW \cap CV$.

$U = BC \cap AD$ so $U = \langle \mathbf{b} + \mathbf{c} \rangle$ since

$$\mathbf{b} + \mathbf{c} = (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{a}.$$

(ii) If U, V, W are collinear then $\mathbf{b} + \mathbf{c}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$ are linearly dependent.

If $x(\mathbf{b} + \mathbf{c}) + y(\mathbf{a} + \mathbf{b}) + z(\mathbf{a} + \mathbf{c}) = \mathbf{0}$ then

$(y + z)\mathbf{a} + (x + y)\mathbf{b} + (x + z)\mathbf{c} = \mathbf{0}$.

Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are linearly independent:

$$\begin{cases} y + z = 0 \\ x + y = 0 \\ x + z = 0 \end{cases}$$

Hence $x = z$ from the first two equations and so $2x = 0$ from the third. If the field is $\mathbb{R}$, this means that $x = 0$, and so $y = z = 0$. Thus $\mathbf{b} + \mathbf{c}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} + \mathbf{c}$ are linearly independent, and so U, V, W are not collinear.

On the other hand, if $F = \mathbb{Z}_2$ then

$(\mathbf{b} + \mathbf{c}) + (\mathbf{a} + \mathbf{b}) + (\mathbf{a} + \mathbf{c}) = 2(\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{0}$ and so U, V and W are collinear.

If $F = \mathbb{Z}_3$ then U, V and W are not collinear, as for $\mathbb{R}$.

Exercise 6:

(i) $\mathbf{a} - \mathbf{c} = (\mathbf{p} + \mathbf{a}) - (\mathbf{p} + \mathbf{c})$ and so

$\mathbf{a} - \mathbf{c} \in AC \cap BD = R$.

$\mathbf{p} + \mathbf{a} + \mathbf{c} = (\mathbf{p} + \mathbf{a}) + \mathbf{c} = (\mathbf{p} + \mathbf{c}) + \mathbf{a} \in BC \cap AD = Q$.

(ii) Suppose that P, Q, R are collinear.
Then $\mathbf{p}$, $\mathbf{p} + \mathbf{a} + \mathbf{c}$, $\mathbf{a} - \mathbf{c}$ are linearly dependent.
Suppose $x\mathbf{p} + y(\mathbf{p} + \mathbf{a} + \mathbf{c}) + z(\mathbf{a} - \mathbf{c}) = \mathbf{0}$.
Then $(x + y)\mathbf{p} + (y + z)\mathbf{a} + (y - z)\mathbf{c} = \mathbf{0}$.

Hence: $\begin{cases} x + y = 0 \\ y + z = 0 \\ y - z = 0 \end{cases}$

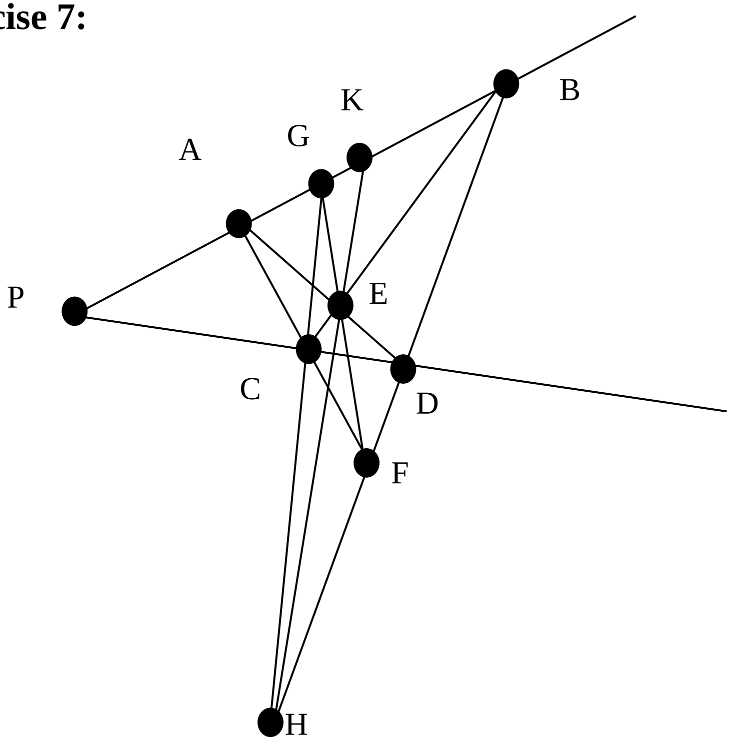
From the last two equations, $2y = 0$ and hence $y = 0$.
It follows that $x = z = 0$, contradicting the linear dependence.

(iii) If $F = \mathbb{Z}_2$ then
$\mathbf{p} + (\mathbf{p} + \mathbf{a} + \mathbf{c}) + (\mathbf{a} - \mathbf{c}) = 2(\mathbf{p} + \mathbf{a}) = \mathbf{0}$
and so P, Q and R are collinear.

(iv) $S = \langle \mathbf{p} + \mathbf{a} - \mathbf{c} \rangle$, $T = \langle \mathbf{p} + 2\mathbf{a} - \mathbf{c} \rangle$ and $U = \langle \mathbf{a} + \mathbf{c} \rangle$.

(v) $2\mathbf{p} + (\mathbf{p} + 2\mathbf{a} - \mathbf{c}) + (\mathbf{a} + \mathbf{c}) = 3(\mathbf{p} + \mathbf{a}) = \mathbf{0}$ if $F = \mathbb{Z}_3$.

Exercise 7:



Let E = AD ∩ BC = $\langle \mathbf{p} + \mathbf{a} + \mathbf{c} \rangle$.
Let F = AC ∩ BD = $\langle \mathbf{a} - \mathbf{c} \rangle$.
Let G = EF ∩ PA = $\langle \mathbf{p} + 2\mathbf{a} \rangle$.
Let H = GC ∩ BD = $\langle \mathbf{p} + 2\mathbf{a} - \mathbf{c} \rangle$.
Let K = HE ∩ PA = $\langle 2\mathbf{p} + 3\mathbf{a} \rangle = \langle \mathbf{p} + (3/2)\mathbf{a} \rangle$.

Exercise 8:
Let E = AD ∩ BC = $\langle \mathbf{p} + \mathbf{a} + \mathbf{c} \rangle$.
Let F = AC ∩ BD = $\langle \mathbf{a} + 4\mathbf{c} \rangle$.
Let G = EF ∩ PA = $\langle \mathbf{p} + 2\mathbf{a} \rangle$.
Let H = GC ∩ BD = $\langle \mathbf{p} + 2\mathbf{a} + 4\mathbf{c} \rangle$.

Let K = HE $\cap$ PA = $\langle 2\mathbf{p} + 3\mathbf{a} \rangle = \langle \mathbf{p} + 4\mathbf{a} \rangle$.
Let L = KC $\cap$ BD = $\langle \mathbf{p} + 4\mathbf{a} + 2\mathbf{c} \rangle$.
Let M = LE $\cap$ PA = $\langle \mathbf{p} + 3\mathbf{a} \rangle$.